Nonlinear QED and Physical Lorentz Invariance

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Abstract

The spontaneous breakdown of 4-dimensional Lorentz invariance in the framework of QED with the nonlinear vector potential constraint $A_\mu^2 = M^2$ (where M is a proposed scale of the Lorentz violation) is shown to manifest itself only as some noncovariant gauge choice in the otherwise gauge invariant (and Lorentz invariant) electromagnetic theory. All the contributions to the photon-photon, photon-fermion and fermion-fermion interactions violating the physical Lorentz invariance happen to be exactly cancelled with each other in the manner observed by Nambu a long ago for the simplest tree-order diagrams - the fact which we extend now to the one-loop approximation and for both the time-like ($M^2 > 0$) and space-like ($M^2 < 0$) Lorentz viola tion. The way how to reach the physical breaking of the Lorentz invariance in the pure QED case (and beyond) treated in the flat Minkowskian space-time is also discussed in some detail.

1 Introduction

Spontaneous violation of Lorentz invariance has attracted considerable attention in the last years as an interesting phenomenological possibility appearing in the framework of various quantum field and string theories [1, 2, 3, 4]. For spontaneous Lorentz invariance violation (LIV), the situation is in some sense similar to the internal symmetry breaking with the corresponding massless Nambu-Goldstone modes appeared. For the LIV such modes are believed to be photons or even non-Abelian gauge fields [5], if the starting symmetry in the Lagrangian is properly chosen.

The handy theoretical laboratory for these considerations happens to be some simple class of the Lagrangian models for the starting massive vector field A_μ where, in one way or another, the nonlinear dynamical constraint of type

$$A_\mu^2 = M^2 \tag{1}$$

(M is a proposed scale of the LIV) is appeared. This constraint means in essence the vector field A_μ develops the vacuum expectation value (VEV) and Lorentz symmetry $SO(1,3)$ formally breaks down to $SO(3)$ or $SO(1,2)$ depending on the sign of the M^2. Such models, which is often called 'bumblebee models' in the literature[1], were introduced by Dirac[6] in the fifties (though in a different context) and then from the LIV point of view was studied by Nambu [7] (see also[8]) independently of the dynamical mechanism which causes the spontaneous Lorentz violation. For this purpose he applied the technique of nonlinear symmetry realizations which appeared successful in handling the spontaneous breakdown of chiral symmetry, particularly, as it appears in the nonlinear σ model[9]. It was shown, while only in the tree approximation and for the time-like LIV ($M^2 > 0$), that the nonlinear constraint (1) implemented into standard QED Lagrangian containing the charged (e) fermion $\psi(x)$

$$\mathcal{L}_{QED} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\gamma\partial + m)\psi - eA_\mu\overline{\psi}\gamma^\mu\psi \tag{2}$$

as some supplementary condition appears in fact as a possible gauge choice which amounts to a temporal gauge for the superlarge (as it is intuitively expected) LIV scale M. At the same time, the S-matrix remains unaltered under such a gauge convention. This particular gauge allows one to interpret QED in terms of the spontaneous LIV with the VEV of vector field of the type $< A_\mu >_0 = (M, 0, 0, 0)$. The LIV, however, is proved to be superficial as it affects only the gauge of vector potential A_μ at least in the tree approximation [7].

In this connection it is a matter of great importance to know whether the Nambu's observation remains when quantum corrections are included into Lagrangian (2). One might think that the tree LIV diagrams are actually cancelled since this level corresponds in fact to the classical theory where the constraint (1) manifests itself as a pure gauge. However, including into play the loop diagrams, which means that one comes to the quantum theory where the vector field canonical commutators introduced (being the non-trivial constraints by themselves), could not allow to consider further the constraint $A_\mu^2 = M^2$ as a gauge choice and, as result, the physically observable LIV effects might appear.

We are focused here on the lowest order LIV processes in QED with the nonlinear dynamical constraint (1) for both of cases of the time-like ($M^2 > 0$) and space-like ($M^2 > 0$) LIV. We explicitly show that for tree approximation all the LIV contributions are exactly cancelled with each other just in a manner which was observed by Nambu a long ago. We then extend our consideration to the calculation of the one-loop LIV contributions to the photon-photon, photon-fermion and fermion-fermion scattering. All these contributions are shown to be mutually cancelled in the framework of the particular dimensional regularization scheme taken (in the way as this scheme is usually applied to QED in noncovariant gauges[10]). This means that the constraint $A_\mu^2 = M^2$ having been treated as the nonlinear gauge choice at a tree (classical) level remains as a gauge condition when quantum effects are taken into account as well. So, in accordance with Nambu's original conjecture one can conclude that the physical Lorentz invariance is left intact at least in the one-loop approximation provided we consider the standard QED Lagrangian (2 (with its gauge invariant $F_{\mu\nu}F^{\mu\nu}$ kinetic term and minimal photon-fermion coupling) taken in the flat Minkowskian space-time.

The paper is organized in the following way. We consider first the non-linear QED Lagrangian (Sec.2) appeared once the dynamical constraint (1) is explicitly implemented into Lagrangian (2), and derive the general Feynman rules for the basic photon-photon and photon-fermion interactions depending no on the particular case of the time-like or space-like LIV. The model appears in essence two-parametric containing the electric charge e and inverse LIV scale $1/M$ as the perturbation parameters so that the LIV interactions are always proportional some powers of them. Then in Sec.3 the LIV tree processes are discussed and, as a typical example, the Lorentz violating Compton effect in the lowest e/M order is considered in detail. In addition to Nambu's conclusion, we have shown that the total cancellation of the physical LIV tree effects has place in both of cases $M^2 > 0$ and $M^2 < 0$. In Sec.4 we present the detailed calculation of the one-loop contributions to the fermion-fermion scattering in the e^3/M order and also briefly discuss the other leading one-loop contributions to the photon-photon, photon-fermion and fermion-fermion scattering up to the next LIV order $1/M^2$. All these effects appear in fact vanishing. Actually, their matrix elements, when they do not vanish by themselves, amount to the differences between pairs of the similar integrals whose integration variables are shifted relative to each other by some constants (being in general arbitrary functions of the external four-momenta of the particles involved) that in the framework of the dimensional regularization leads to their total cancellation. And, finally, we give our conclusions in Sec.5. Among them the way how to reach the physical breaking of Lorentz invariance in the flat Minkowskian space-time is also discussed in some detail.

2 The Lagrangian and Feynman rules

2.1 The Lagrangian

We consider simultaneously both of the above-mentioned LIV cases, time-like or space-like, introducing some unit vector n_μ ($n_\mu^2 \equiv n^2 = \pm 1$ depending on the sign of M^2, respectively) so as to have the following general parametrization for the vector potential A_μ

in the Lagrangian (2) of the type

$$A_\mu = a_\mu + \frac{n_\mu}{n^2}(n \cdot A) \tag{3}$$

where the a_μ is pure Goldstonic mode

$$n \cdot a = 0 \tag{4}$$

while the Higgs mode (or the A_μ component in the vacuum direction) is given by the scalar product $n \cdot A$. Substituting this parametrization into the vector field constraint (1) one comes to the equation for $n \cdot A$ (taking, for simplicity, the positive sign for the square root only)

$$n \cdot A = \left[n^2(M^2 - a_\nu^2)\right]^{\frac{1}{2}} \tag{5}$$

which for the particular time-like ($M^2 > 0$, $n^2 = 1$) and space-like ($M^2 < 0$, $n^2 = -1$) VEV cases takes the simpler forms

$$A_0 = \left[M^2 + a_i^2)\right]^{\frac{1}{2}}, \quad n_\mu = (1,0,0,0), \quad a_\mu = (0, A_i), \quad (i = 1,2,3) \tag{6}$$

and

$$A_3 = \left[\left|M^2\right| + a_\beta^2\right]^{\frac{1}{2}}, \quad n_\mu = (0,0,0,1), \quad a_\mu = (A_\beta, 0), \quad (\beta = 0,1,2) \tag{7}$$

respectively (for the space-like case the vacuum direction was chosen along the third axis). For the high LIV scale M, as is expected, the equation for $n \cdot A$ (5) can be then expanded in powers of $\frac{a_\nu^2}{M^2}$

$$n \cdot A = M - \frac{n^2 a_\nu^2}{2M} + O(1/M^2) \tag{8}$$

where M is defined always positive, while $n^2 \equiv n_\mu^2 = n_0^2 - n_i^2$ and $a_\nu^2 = a_0^2 - a_i^2$ are determined according their non-zero components given in Eqs. (6) and (7) for particular cases.

We proceed further putting that new parametrization (3) into our basic Lagrangian (2), using then the above expansion for the Higgs mode $n \cdot A$ (8) and making the appropriate redefinition of fermion field ψ according to

$$\psi \rightarrow e^{ieM(n \cdot x)}\psi \tag{9}$$

so that the mass-type term $eM\overline{\psi}(\gamma \cdot n)\psi$ appearing from the expansion of the fermion current interaction in the Lagrangian (2) will be exactly cancelled by an analogous term stemming now from the fermion kinetic term. So, we eventually arrive at the Lagrangian for the a_μ field (denoting its strength tensor by $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$)

$$\begin{aligned}
\mathcal{L}_{NL} = & -\frac{1}{4}f_{\mu\nu}f^{\mu\nu} + \overline{\psi}(i\gamma\partial + m)\psi - ea_\mu\overline{\psi}\gamma^\mu\psi + \\
& -\frac{1}{4M}f_{\mu\nu}\left[(n^\mu\partial^\nu - n^\nu\partial^\mu)a_\rho^2 n_\sigma^2 + \cdots\right] - \frac{1}{16M^2}\left[(n^\mu\partial^\nu - n^\nu\partial^\mu)a_\rho^2 + \cdots\right]^2 + \\
& + e\frac{a_\rho^2 n_\sigma^2}{2M}(1 + \cdots)\overline{\psi}(\gamma \cdot n)\overline{\psi}
\end{aligned} \tag{10}$$

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where we collected the linear and nonlinear (in the a_μ fields) terms separately leaving only terms corresponding to the expansion in the Higgs mode $n \cdot A$, as is taken in Eq.(8), and also retained the former notation for fermion ψ. We take the Greek letters for the Lorentz indices ($\mu, \nu, \rho, \sigma = 0, 1, 2, 3$) and the metric is $g_{\mu\nu} = (1, -1, -1, -1)$, while everywhere when appears $(n^2)^2$ (and higher powers of n^2) we replace it by 1. For the photon-electron and photon-photon interactions it follows then in the lowest approximation

$$\mathcal{L}_{NL}^{int} = -ea_\mu \overline{\psi} \gamma^\mu \psi + e\frac{n^2 a_\rho^2}{2M} \bar{\psi}(\gamma \cdot n)\psi - \frac{n^2}{M}(\partial_\mu a_\nu n^\mu a_\rho \partial^\nu a^\rho) - \frac{1}{16M^2}\left[(n^\mu \partial^\nu - n^\nu \partial^\mu)a_\rho^2\right]^2 \tag{11}$$

The Lagrangian (10) together with the gauge fixing condition (4) completes the nonlinear σ model type construction for quantum electrodynamics. We call this the nonlinear QED. The model contains the massless vector Goldstone boson modes and keeps the massive Higgs mode frozen, and in the limit $M \to \infty$ the model (given just by the first line in the Lagrangian $\mathcal{L}_{NL}$ (10)) is indistinguishable from conventional QED taken in the temporal or axial gauge (4). So, for this part of the Lagrangian $\mathcal{L}_{NL}$ the spontaneous LIV only means the noncovariant gauge choice (4) in otherwise the gauge invariant (and Lorentz invariant) theory. However, we will show in the next section that also all other terms in the $\mathcal{L}_{NL}$ (10), though being by themselves the Lorentz and $C(CPT)$ violating ones, cause no the physical LIV effects at least in the one-loop approximation.

2.2 The Feynman rules

They for the interaction Lagrangian $\mathcal{L}_{NL}^{int}$ (11) include:

i/ An ordinary QED photon-electron vertex is

$$-ie\gamma_\mu \tag{12}$$

ii/ The contact 2-photon-electron vertex is given by

$$i\frac{eg_{\mu\nu}n^2}{M}(\gamma \cdot n) \tag{13}$$

iii/ The 3-photon vertex (with photon 4-momenta k_1, k_2 and k_3) is appeared as

$$-\frac{in^2}{M}\left[(k_1 \cdot n)k_{1,\alpha}g_{\beta\gamma} + (k_2 \cdot n)k_{2,\beta}g_{\alpha\gamma} + (k_3 \cdot n)k_{3,\gamma}g_{\alpha\beta}\right] \tag{14}$$

where the second index in the each momentum k_1, k_2 and k_3 denotes its Lorentz component;

iv/ The 4-photon vertex (with photon 4-momenta k_1, k_2, k_3 and k_4) is

$$\frac{-i}{M^2}\{[(n^2(k_1 + k_2)^2 - (n(k_1 + k_2))^2]g_{\alpha\beta}g_{\mu\nu} + [n^2(k_1 + k_3)^2 - (n(k_1 + k_3))^2]g_{\alpha\mu}g_{\beta\nu} +$$
$$+[n^2(k_1 + k_4)^2 - (n(k_1 + k_4))^2]g_{\alpha\nu}g_{\beta\mu}\} \tag{15}$$

v/ The photon propagator is in general (for $n^\mu a_\mu = 0$)

$$D_{\mu\nu}(k) = \frac{-i}{k^2 + i\epsilon}\left(g_{\mu\nu} - \frac{n_\mu k_\nu + k_\mu n_\nu}{n \cdot k} + \frac{n^2 k_\mu k_\nu}{(n \cdot k)^2}\right) \tag{16}$$

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(where k^2 stands for the photon 4-momentum squared) being automatically satisfied the orthogonality condition $n^\mu D_{\mu\nu}(k) = 0$ and on-shell transversality $k_\mu D_{\mu\nu}(k) = 0 \quad (k^2 = 0)$. The latter means that the free photon with the polarization vector $\varepsilon^\mu(k, k^2 = 0)$ is always appeared transverse $k_\mu \varepsilon^\mu(k) = 0$.

vi/ The electron propagator (standard) is

$$S(p) = \frac{i}{\gamma \cdot p - m} \tag{17}$$

3 The tree LIV contribution: photon-fermion scattering

We start with a calculation of the tree LIV contributions to the photon-fermion scattering. We show now that such contributions to the standard Compton effect taken in the lowest e/M order are exactly cancelled for any choice of the constant vector n^μ (or for the time-like or the space-like LIV). These contributions are given by two diagrams (see Fig.1).
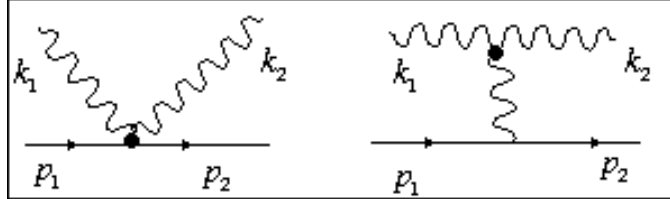

Figure 1: The LIV contribution to the Compton effect in the tree approximation (in the order of $O(\frac{e}{M})$)

3.1 The LIV matrix element

This matrix element $i\mathcal{M}$ corresponding to these two diagrams is given by

$$i\mathcal{M} = [\bar{u}(p_2)(p_2)O_{\mu\nu}u(p_1)] \cdot \varepsilon^\mu(k_1)\varepsilon^\nu(k_2) \tag{18}$$

where the ingoing and outgoing electron spinors $u(p_1)$ and $u(p_2)$ (with momenta p_1 and p_2) and photon polarization vectors $\varepsilon_\mu(k_1)$ and $\varepsilon_\nu(k_2)$ (with momenta k_1 and k_2) are explicitly

indicated. The $O^{\mu\nu}$ consists of sum of both of diagrams and is in fact

$$
\begin{aligned}
O_{\mu\nu} &= i\frac{eg_{\mu\nu}n^2}{M}(\gamma n) - \frac{in^2}{M}[(k\cdot n)k_\beta g_{\mu\nu} + (k_2 n)k_{2,\nu}g_{\mu\beta} + (k_1 n)k_{1,\mu}g_{\beta\nu})] \cdot \\
&\quad \cdot(-ie\gamma_\alpha)\frac{(-i)}{k^2}\left(g^{\alpha\beta} - \frac{n^\alpha k^\beta + k^\alpha n^\beta}{n\cdot k} + \frac{n^2 k^\alpha k^\beta}{(n\cdot k)^2}\right)
\end{aligned}
\tag{19}
$$

where k is the transferred 4-momentum $k = p_2 - p_1 = k_1 - k_2$.

3.2 Cancellation of the tree LIV contributions

Since the ingoing and outgoing photons appear transverse ($k_{1,2}\cdot\varepsilon(k_{1,2}) = 0$) there are only left the terms

$$
i\mathcal{M} = i\frac{en^2 g_{\mu\nu}}{M}\bar{u}(p_2)\left[(\gamma\cdot n) + \frac{(kn)k_\beta\gamma_\alpha}{k^2}\left(g^{\alpha\beta} - \frac{n^\alpha k^\beta + k^\alpha n^\beta}{(n\cdot k)} + \frac{n^2 k^\alpha k^\beta}{(n\cdot k)^2}\right)\right]u(p_1)\cdot\varepsilon^\mu(k_1)\varepsilon^\nu(k_2)
\tag{20}
$$

in the matrix element $i\mathcal{M}$. So, after the evident simplification in the square bracket

$$
(\gamma\cdot n) + \frac{(k\cdot n)k^\alpha\gamma_\alpha}{k^2} + \frac{n^2 k^\alpha\gamma_\alpha}{(n\cdot k)} - (\gamma n) - \frac{(kn)k^\alpha\gamma_\alpha}{k^2} = \frac{n^2\hat{k}}{(n\cdot k)}
\tag{21}
$$

one is finally led to the matrix element ($k = p_2 - p_1$)

$$
i\mathcal{M} = i\frac{en^2}{M}\left[\bar{\psi}(p_2)\frac{\hat{p}_2 - \hat{p}_1}{n\cdot(p_2 - p_1)}\psi(p_1)\right]\cdot[\varepsilon(k_1)\cdot\varepsilon(k_2)]
\tag{22}
$$

which unavoidably amounts to zero due to the fermionic current conservation

$$
\bar{u}(p_2)(\hat{p}_2 - \hat{p}_1)u(p_1) = 0
\tag{23}
$$

3.3 The other tree LIV processes

We have also considered the other processes in the tree approximation, such as the pure photon-photon scattering (going through the pole 3-photon and contact 4-photon diagrams), electron-electron scattering with emission of extra photon ($e + e \to e + e + \gamma$) etc. taken in the lowest order, and everywhere the LIV contributions are completely cancelled. Moreover, in addition to the Nambu's conclusion we found that such a cancellation has place for both of signs of M^2, as one can readily see from the above-considered Compton scattering case (i.e. the cancellation occurs for any choice of the vector n_μ). It seems very likely that such tendency remains in the higher-order tree diagrams as well, since there works the special mechanism of cancellation between the 3-photon diagram and the corresponding contact diagram (like as we explicitly showed for the Compton scattering diagrams). Remarkably, the same mechanism of cancellation happens to also work for the loop diagrams, as we can see in the next section.

4 The loop LIV contribution: fermion-fermion scattering

Consideration of the loop LIV contributions appears much more complicated since the non-linear QED (10) seems to be (at least formally) nonrenormalizable theory in which even the one-loop divergences could be gauge dependent. In this connection the choice of an adequate regularization scheme in the model is a matter of a crucial importance. As the typical process including the one-loop LIV corrections we consider in detail the ee' scattering process in the lowest e^3/M order and also briefly discussed the proper one-loop contributions to the photon-photon, photon-fermion and fermion-fermion scattering up to the higher LIV order $1/M^2$. The e' could be any other lepton, say, muon μ or taon τ, so that the complications related with the identical fermions are avoided. We show here the LIV cancellation mechanism, which appeared so effective in the above for the tree LIV diagrams, happens to work for the loop contributions as well in the framework of the dimensional regularization scheme taken. In that scheme the possible surface terms appearing from the (linearly and higher) divergent integrals in the model automatically vanish thus allowing the LIV cancellation mechanism to work without serious consequences. At the same time one could apply some other regularization which would feel such surface terms and, as a result, some surviving physical LIV effects could appear. We discuss this point for the ee' scattering process in detail at the end of this section

4.1 Basic diagrams and matrix element

The basic LIV diagrams for the ee' scattering stem from the interaction Lagrangian (10) properly extended to include the lepton e' as well. There are in fact eight possible diagrams in the lowest order e^3/M, as are given in the Fig.2. According to them and Feynman rules formulated in Sec.2.2 one immediately finds the corresponding matrix element. Actually, we consider four diagrams (1+2+3+4) (the contribution of the other four diagrams (5+6+7+8) follows from the simple replacement of 4-momenta $p \Leftrightarrow p'$ and $k \Leftrightarrow k'$ and masses $m \Leftrightarrow m'$ in the matrix element $i\mathcal{M}_{1-4}$). Considering first the diagrams (1+2) one has

$$
\begin{aligned}
i\mathcal{M}_{1+2} = \quad & \bar{u}(k')\{ \ i\frac{eg_{\alpha\beta}n^2}{M}(\gamma \cdot n) + \\
& +(-ie\gamma_\lambda)D^{\lambda\rho}(p-k)(\frac{-in^2}{M})[[(p-k)\cdot n](p-k)_\rho g_{\alpha\beta} + \\
& +[(p-q)\cdot n](p-q)_\alpha g_{\beta\rho} + [(k-q)\cdot n](k-q)_\beta g_{\alpha\rho}]\}u(p') \cdot \\
& \cdot[\bar{u}(k)D^{\beta\nu}(k-q)(-ie\gamma_\nu)(i)\frac{\hat{q}+m}{q^2-m^2}D^{\alpha\mu}(p-q)(-ie\gamma_\mu)u(p)]
\end{aligned} \tag{24}
$$

where the total energy-momentum conservation $p+p\prime = k+k\prime$ (so that $D^{\lambda\rho}(p-k) = D^{\lambda\rho}(p\prime - k\prime)$ etc.), as well as the standard integration with respect to the internal 4-momentum q ($i\mathcal{M}_{1+2} \to \int \frac{d^4q}{(2\pi)^4}(i\mathcal{M}_{1+2})$) are also implied.
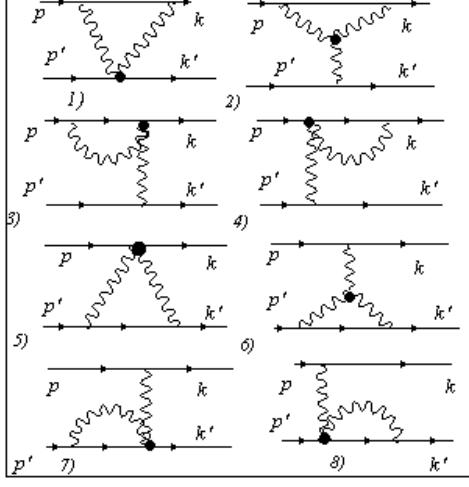
Figure 2: The LIV contribution to the $e - e'$ scattering in the one-loop approximation (in the order of $O(\frac{e^3}{M})$)

4.2 Cancellation of the loop LIV contributions

One can see now that the contact $\gamma \cdot n$ term in $i\mathcal{M}_{1+2}$ is cancelled with the first term in the square bracket (containing the 3-photon vertex terms) since the sum of these two terms can be rewritten as (using the photon propagator form (16) for $D^{\lambda\rho}(p-k)$)

$$i\frac{eg_{\alpha\beta}n^2}{M}\bar{u}(k')\{(\gamma \cdot n) + ie\gamma_\lambda D^{\lambda\rho}(p-k)[(p-k)\cdot n](p-k)_\rho\}u(p')) = \tag{25}$$

$$i\frac{eg_{\alpha\beta}n^2}{M}\bar{u}(k')\{(\gamma \cdot n) + \frac{[(p-k)\cdot n](\hat{p}-\hat{k})}{(p-k)^2} + \frac{n^2(\hat{p}-\hat{k})}{(p-k)\cdot n} - (\gamma \cdot n) - \frac{((p-k)\cdot n)(\hat{p}-\hat{k})}{(p-k)^2}\}u(p')$$

$$= \quad i\frac{eg_{\alpha\beta}}{M}\bar{u}(k')\frac{\hat{p}-\hat{k}}{(p-k)\cdot n}u(p')$$

which due to the conservation of fermion current (and 4-momentum conservation $p - k = -(p' - k')$) certainly comes to zero. This is just the general mechanism already found in the Compton effect (see Sec.3): the contact 2-photon diagram contribution is always cancelled with the part of the pole 3-photon diagram contribution which is free from the internal integration.

At the same time its other parts still remain. They correspond just to two survived

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3-photon vertex terms in the $i\mathcal{M}_{1+2}$ which are subject to an integration with respect to the internal 4-momentum q. They amount to (properly rewritten)

$$i\mathcal{M}_{1+2} = \frac{ie^3n^2}{M}[\bar{u}(k')\gamma_\lambda u(p')]D^{\lambda\rho}(p-k)\{[(p-q)\cdot n](p-q)_\alpha D^{\alpha\mu}(p-q)g_{\beta\rho}D^{\beta\nu}(k-q) +$$

$$+[(k-q)\cdot n](k-q)_\beta D^{\beta\nu}(k-q)g_{\alpha\rho}D^{\alpha\mu}(p-q)\}[\bar{u}(k)\gamma_\nu\frac{\hat{q}+m}{q^2-m^2}\gamma_\mu u(p)] \qquad (26)$$

Using there

$$[(p-q)\cdot n](p-q)_\alpha D^{\alpha\mu}(p-q) = (-i)[\frac{n^2(p-q)^\mu}{(p-q)\cdot n}-n^\mu] \qquad (27)$$

$$[(k-q)\cdot n](k-q)_\beta D^{\beta\nu}(k-q) = (-i)[\frac{n^2(k-q)^\nu}{(k-q)\cdot n}-n^\nu]$$

and then (using Dirac equations $\hat{p}\cdot u(p)=m\cdot u(p)$, $\bar{u}(k)\cdot\hat{k}=\bar{u}(k)\cdot m$ in the fermionic sector)

$$(p-q)^\mu[\bar{u}(k)\gamma_\nu\frac{\hat{q}+m}{q^2-m^2}\gamma_\mu u(p)] = -\bar{u}(k)\gamma_\nu u(p) \qquad (28)$$

$$(k-q)^\nu[\bar{u}(k)\gamma_\nu\frac{\hat{q}+m}{q^2-m^2}\gamma_\mu u(p)] = -\bar{u}(k)\gamma_\mu u(p)$$

one is finally led to

$$i\mathcal{M}_{1+2} = \frac{e^3n^2}{M}[\bar{u}(k')\gamma_\lambda u(p')][\bar{u}(k)\gamma_\mu u(p)]D^{\lambda\rho}(p-k)D^\mu_\rho(k-q)\frac{1}{(q-p)\cdot n} +$$

$$+\frac{e^3n^2}{M}[\bar{u}(k')\gamma_\lambda u(p')][\bar{u}(k)\gamma_\mu u(p)]D^{\lambda\rho}(p-k)D^\mu_\rho(p-q)\frac{1}{(q-k)\cdot n} -$$

$$-\frac{e^3n^2}{M}[\bar{u}(k')\gamma_\lambda u(p')][\bar{u}(k)\gamma_\mu\frac{\hat{q}+m}{q^2-m^2}(\gamma\cdot n)u(p)]D^{\lambda\rho}(p-k)D^\mu_\rho(k-q) - (29)$$

$$-\frac{e^3n^2}{M}[\bar{u}(k')\gamma_\lambda u(p')][\bar{u}(k)(\gamma\cdot n)\frac{\hat{q}+m}{q^2-m^2}\gamma_\mu u(p)]D^{\lambda\rho}(p-k)D^\mu_\rho(p-q)$$

where $D^\mu_\rho(k-q)=g_{\alpha\rho}D^{\alpha\mu}(k-q)$ etc.

Let us turn to the diagrams (3) and (4) in the Fig.2. Their calculation goes faster and one readily has

$$i\mathcal{M}_{3+4} = \bar{u}(k')(-ie\gamma_\lambda)D^{\lambda\rho}(p-k)u(p')\cdot \qquad (30)$$

$$\cdot\bar{u}(k)[(-ie\gamma_\mu)(i)\frac{\hat{q}+m}{q^2-m^2}D^{\mu\alpha}(k-q)(i\frac{eg_{\alpha\rho}n^2}{M})(\gamma\cdot n) +$$

$$+(i\frac{eg_{\alpha\rho}n^2}{M})(\gamma\cdot n)D^{\mu\alpha}(k-q)(i)\frac{\hat{q}+m}{q^2-m^2}(-ie\gamma_\mu)]u(p)$$

$$= \frac{e^3n^2}{M}[\bar{u}(k')\gamma_\lambda u(p')][\bar{u}(k)\gamma_\nu\frac{\hat{q}+m}{q^2-m^2}(\gamma\cdot n)u(p)]D^{\lambda\rho}(p-k)D^\nu_\rho(k-q) +$$

$$+\frac{e^3n^2}{M}[\bar{u}(k')\gamma_\lambda u(p')][\bar{u}(k)(\gamma\cdot n)\frac{\hat{q}+m}{q^2-m^2}\gamma_\mu u(p)]D^{\lambda\rho}(p-k)D^\mu_\rho(p-q)$$

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One can immediately see that the $i\mathcal{M}_{3+4}$ is completely cancelled with the last two terms in the $i\mathcal{M}_{1+2}$ (29). Strictly speaking these terms in the $i\mathcal{M}_{3+4}$ (30) can differ (as being followed from the different diagrams) from the corresponding terms in the $i\mathcal{M}_{1+2}$ by some arbitrary shifts in the integration variable q. One could take instead, say, $q+a$ and $q+b$ in the integrals in $i\mathcal{M}_{3+4}$ where a and b can be some arbitrary function of the external momenta p, k, p' and k'. In general, this would lead to some finite surface terms for the difference of the linearly divergent integrals in the $i\mathcal{M}_{1+2}$ and $i\mathcal{M}_{3+4}$ as they actually are (see some discussion below). However, in the dimensional regularization scheme taken here such surface terms automatically vanish. So, the total contribution of four diagrams (1+2+3+4) eventually comes to

$$
\begin{aligned}
i\mathcal{M}_{1-4} \;=\; & \frac{e^3 n^2}{M}[\bar{u}(k')\gamma_\lambda u(p') \cdot \bar{u}(k)\gamma_\mu u(p)]D^{\lambda\rho}(p-k)\cdot \\
& \cdot \int \frac{d^4q}{(2\pi)^4}\left(\frac{D^\mu_\rho(q-k)}{(q-p)\cdot n}+\frac{D^\mu_\rho(q-p)}{(q-k)\cdot n}\right)
\end{aligned}
\tag{31}
$$

where we explicitly indicated the integration with respect to the internal 4-momentum q (and used the symmetry of the propagator $D^\mu_\rho(k-q)=D^\mu_\rho(q-k)$). One is allowed then to change internal momentum q to $-q$ in the second term in the integral in (31) and rewrite it as

$$
I^\mu_\rho(p,k) = \int \frac{d^4q}{(2\pi)^4}\left(\frac{D^\mu_\rho(q-k)}{(q-p)\cdot n}-\frac{D^\mu_\rho(q+(p+k)-k)}{(q+(p+k)-k)\cdot n}\right)
\tag{32}
$$

so that one has (again) the difference of two similar integrals in the $I^\mu_\rho(p,k)$ which only differs by the integration variables q and $q+(p+k)$, respectively, that in the framework of the dimensional regularization leads to their total cancellation.

Therefore, we have shown that the total matrix element $i\mathcal{M}_{tot}$ for the electron LIV scattering taken in the lowest e^3/M order including contribution of all eight diagrams (1-8) and given, as was said in the above, by an extension

$$
i\mathcal{M}_{tot} = i\mathcal{M}_{1-4} + (p \Leftrightarrow p', \;\; k \Leftrightarrow k', \;\; m \Leftrightarrow m')
\tag{33}
$$

does finally vanish in the dimensional regularization scheme.

4.3 Integration: the surface term problem

In conclusion, it seems interesting to discuss this LIV cancellation mechanism from the surface term point of view in more detail, particularly, for the integral $I^\mu_\rho(p,k)$ (32). What the physical LIV effect might be expected if the corresponding surface term were survived? Note that, in the contrast to the above case with the $i\mathcal{M}_{3+4}$, the shift in the integration variable in the $I^\mu_\rho(p,k)$ is completely determined (as $(p+k)$) since both of integrals in it follow from the same diagram (2) in the Fig.2. These integrals would, of course, exactly cancel each other by the proper shift of the variables, if they were finite. However, they, as one can readily see, are linearly divergent and, generally speaking, such a shift of the integration variables is not allowed. Instead, one should calculate the surface term in the $I^\mu_\rho(p,k)$, as one usually does it when calculating the triangle anomaly diagrams. And exactly, as in the anomaly case, this surface term appears finite.

So, using the Gauss theorem one comes to

$$
\begin{aligned}
I_\rho^\mu(p,k) &= -\frac{(p+k)_\tau}{(2\pi)^4} \int d^4q \frac{\partial}{\partial q^\tau}\left(\frac{D_\rho^\mu(q-k)}{(q-p)\cdot n}\right) = \\
&= -\frac{(p+k)_\tau}{(2\pi)^4}(2i\pi^2)\lim_{q\to\infty} q^\tau q^2 \left(\frac{-i(g_\rho^\mu - \frac{n^\mu(q-k)_\rho + n_\rho(q-k)^\mu}{n\cdot(q-k)} + \frac{n^2(q-k)^\mu(q-k)_\rho}{[n\cdot(q-k)]^2})}{(q-p)^2(q-p)\cdot n}\right)
\end{aligned}
\tag{34}
$$

Now neglecting everywhere all the terms of the order $O(p/q, k/q)$ and using for the limiting values of q the evident equalities:

$$
\frac{q^\mu q_\rho}{q^2} = \frac{1}{4}g_\rho^\mu, \quad \frac{q^\mu q_\rho}{(n\cdot q)^2} = g_\rho^\mu \frac{1}{n^2}, \quad \frac{n^\mu q_\rho + n_\rho q^\mu}{n\cdot q} = \frac{2n^\mu n_\rho}{n^2}, \quad \frac{q^\tau}{n\cdot q} = \frac{n^\tau}{n^2}
\tag{35}
$$

one is eventually led to the finite value of the integral

$$
I_\rho^\mu(p,k) = -\frac{(p+k)\cdot n}{4\pi^2 n^2}\left(g_\rho^\mu - \frac{n^\mu n_\rho}{n^2}\right)
\tag{36}
$$

as was expected.

The total matrix element $i\mathcal{M}_{tot}$ is now readily followed using the orthogonality of the propagator $n_\rho D^{\lambda\rho}(p-k) = 0$ so that the last term in the $I_\rho^\mu(p,k)$ is not relevant. Taking also that only the first term in the $D^{\lambda\rho}(p-k)$ contributes when it is sandwiched between conserved fermion currents in the $i\mathcal{M}_{tot}$, and properly replacing momenta $(p \Leftrightarrow p', \ k \Leftrightarrow k')$ to include all the contributions, one finally comes to

$$
i\mathcal{M}_{tot} = i\frac{e^3}{2\pi^2}\frac{(p+p')\cdot n}{M}[\bar{u}(k')\gamma_\mu u(p')\frac{1}{(p-k)^2}\bar{u}(k)\gamma^\mu u(p)]
\tag{37}
$$

where we have also used the total 4-momentum conservation (giving $p+k+p'+k' = 2(p+p')$) when collecting $i\mathcal{M}_{1-4}$ and $i\mathcal{M}_{5-8}$ in the $i\mathcal{M}_{tot}$.

Having the integrated LIV matrix element $i\mathcal{M}_{tot}$ (37) for the ee'-scattering one can apply it (when properly modifying) to any particular case, such as the ee and $e\mu$ scatterings, e^+e^--annihilation, $e^+e^- \to \mu^+\mu^-$ conversion and so on, thus observing in the practical physics all the peculiarities related with the Lorentz symmetry breaking. The common feature for all these processes seems to be the direct dependence on some particular component of the total momentum $(p+p')\cdot n$ which for the case $M^2 > 0$ (fixed in the temporal choice of the vector $n^\mu = (1,0,0,0)$) means the dependence on the total energy of ee'-scattering. Particularly, for the LIV correction to the Coulomb potential stemming from the matrix elements $i\mathcal{M}_{tot}$ (37) in the non-relativistic limit

$$
\begin{aligned}
p &= (m,\vec{p}), \quad k = (m,\vec{k}) \\
p\prime &= (m\prime,\vec{p'}), \quad k = (m\prime,\vec{k}\prime) \\
\bar{u}(k)\gamma_\mu u(p) &= [\bar{u}(m)\gamma_0 u(m), 0, 0, 0]
\end{aligned}
\tag{38}
$$

(taken to the $O(\vec{p}^2, \vec{k}^2, \vec{p'}^2, \vec{k'}^2)$ accuracy) one has

$$
i\mathcal{M}_{tot} = -i\frac{e^3}{2\pi^2}\frac{(m+m')}{M}[\bar{u}(m\prime)\gamma_0 u(m')\frac{1}{|\vec{p}-\vec{k}|^2}\bar{u}(m)\gamma_0 u(m)]
\tag{39}
$$

12

Collecting it with a standard QED matrix element (given by an ordinary one-photon exchange) one is finally led to the total Coulomb potential for the $ee\prime$ interaction

$$V_{el}(r) = \frac{e^2}{4\pi}\frac{1}{r} + \frac{e^3}{(2\pi)^3}\frac{1}{r}(\frac{m+m'}{M}) \tag{40}$$

We have received in the certain sense somewhat exciting LIV extension of the standard Coulomb potential. The extra term in it feels, as one can see, the masses of the scattering particles and, thus, has some gravitational character. At the same time its sign (repulsive or attractive) is determined by the electromagnetism. Very remarkably, for the same sign charges just the 'anti-gravity' holds.

However, such a would-be finite physical LIV result depend, as one can see, on some special condition for the virtual photon and fermion four-momentum running in the loop diagrams in the Fig.2. Such a condition was taken in the above so as to have the total reduction of the diagram (3) and (4) with the proper parts from the diagram (2), while remaining some of its well defined parts (31). This corresponds just to the zero shifts ($a = b = 0$) in the integration variables in the integrals in the matrix element $i\mathcal{M}_{3+4}$ (30) with respect to the integration variable in the integrals in the $i\mathcal{M}_{1+2}$ (29). Another condition could give in principle another effect since the linearly (and higher) divergent integrals are generally gauge and regularization scheme dependent. Point is, however, that there happens to appear a freedom in the case considered to choose the four-momentum running in the loops in a gauge invariant way not to have the LIV at all. This is just what suggests the dimensional regularization scheme.

4.4 The other one-loop LIV contributions

Following to the same way of argumentation we have also considered the other leading one-loop LIV contributions. In the same order e^3/M we have calculated such contributions to the photon-photon and photon-fermion scattering as well. Further, finding no the LIV corrections to photon and fermion propagators we have checked then the proper one-loop contributions to the fermion-fermion and photon-fermion scattering in the next LIV order $1/M^2$. All these effects appears in fact vanishing due to the same cancellation mechanism which we observed above for the tree and loop LIV contributions: cancellation between diagrams containing the 3-photon vertex (where one or two of photons interacts with fermion in an usual QED way) and those containing the contact 2-photon-fermion vertex. Actually, their matrix elements (when they do not vanish by themselves as, say, it takes place for the corrections to the photon and fermion propagators) amount to the differences between pairs of the similar integrals whose integration variables happen to be shifted relative to each other by some constants (being in general arbitrary functions of the external four-momenta of the particles involved) that in the framework of the dimensional regularization leads to their total cancellation.

5   Conclusion

Some concluding remarks are in order:

1/ To the lowest LIV order (in $1/M$) all the tree diagrams caused by the non-linear constraint $A_\mu^2 = M^2$ in photon-photon, photon-fermion, and fermion-fermion scatterings are exactly cancelled. In addition to the Nambu's conclusion, we have shown that it has place for the both type of the LIV, time-like ($M^2 > 0$) or space-like ($M^2 < 0$);

2/ It seems very likely that such tendency remains in the higher-order tree diagrams as well, since there works the special mechanism of cancellation between diagrams containing the 3-photon vertex (where one or two of photons interacts with fermion in an usual QED way) and those containing the contact 2-photon-fermion vertex;

3/ For the one-loop diagrams this cancellation mechanism is also effective. We have explicitly demonstrated it calculating the one-loop contributions to the fermion-fermion scattering in the order of e^3/M and also briefly discussed the proper contributions to the photon-photon, photon-fermion and fermion-fermion scattering up to the higher LIV order $1/M^2$. All these radiative effects appears in fact vanishing in the framework of the dimensional regularization scheme taken;

4/ Together with all that the most important conclusion is that for the pure QED like theories the standard potential-induced spontaneous symmetry breaking (leading to the nonlinear field constraint $A_\mu^2 = M^2$ or to its more familiar linearized form $A_\mu = a_\mu + n_\mu M$) is in fact superficial in the Lorentz symmetry case even in the case when quantum corrections in terms of the one-loop contributions are included into consideration. This happens to correspond only to fixing the gauge of the vector potential in a special manner provided that its kinetic term is taken in the standard gauge invariant $F_{\mu\nu} F^{\mu\nu}$ form.

In this connection a critical question may appear in which extent our basic conclusion related with the vanishing of all the one-loop LIV contributions is the dimensional regularization scheme dependent, which was used in the paper. We have shown in Sec.4.3 in somewhat provocative way how important LIV modification of the Coulomb law could follow if some special condition for the virtual photon and fermion four-momentum running in the loop diagrams in the Fig.2. were taken. The point is, however, such a condition would require another regularization scheme which could violate gauge invariance in the theory and that scheme, therefore, could not be implemented at higher loops to keep the theory unitary. The certain advantage of the dimensional regularization scheme over the other ones is that this scheme, preserving by itself the gauge invariance in the theory, gives the direct and undistorted answer to the basic question we are interested here - is the nonlinear field constraint $A_\mu^2 = M^2$ only the gauge choice leading no to the observable Lorentz violating effects in the otherwise gauge and relativistically invariant QED? And this answer is positive at least in the one-loop approximation.

Nonetheless, whether it is possible to have the physical Lorentz violation in some minimal extension of the conventional QED model. One way, proposed Nambu[7], is, in fact, to add a term of type $\beta(\partial_\mu A^\mu)^2$ in the QED Lagrangian (2) which would lead to the actual LIV in our nonlinear QED with three massless Goldstonic modes appeared, two transversal and one longitudinal. This way, however, leads to the uncontrollably large Lorentz violation since there is no reason to consider the constant β in the above bilinear term to be acceptably small.

Another way was suggested recently in the paper[11]. It was shown that starting with a general massive vector field theory one naturally comes to the Lagrangian (2) with the nonlinear field constraint (1) if the pure spin-1 value for the vector field $A_\mu(x)$ is required. In essence, this model differs from the model considered in Sec.2 only in one substantial

respect - the vector potential $A_\mu(x)$ appears automatically satisfying to the transversality condition $\partial_\mu A^\mu = 0$. However, it happens enough for the physical LIV to occur. Actually, as now one can see, due to the transversality condition the above LIV cancellation mechanism is no longer working: the 3-photon (and generally odd-number-photon) vertex disappears from theory so that the contact 2-photon-fermion (and even-number-photon-fermion in general) vertex diagram contributions being proportional to the powers of the $1/M$ are left uncompensated. This seems to be the only possible way how one could reach the small and controllable physical breaking of Lorentz invariance in the pure QED taken in the flat Minkowskian space-time. Unfortunately, such a theory having now two field constraints appears in fact too restrictive in a sense that the standard Coulomb law for the time-like LIV ($M^2 > 0$) becomes problematic and should independently be introduced as a generic four-fermion interaction. So, generally the LIV inspired QED with its strictly conserved fermion current comes to the total conversion of the LIV into gauge degrees of freedom of the massless photon thus coinciding with a conventional QED. However, the situation is drastically changed when the internal symmetry, together with the Lorentz invariance, is spontaneously broken, as it takes place in Standard Model and Grand Unified Theories - the LIV becomes physically observable[12].

Acknowledgments

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